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Mappings and Accuracy for Chebyshev Pseudo-spectral Approximations*

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The effect of mappings on the approximation, by Chebyshev collocation, of functions which exhibit localized regions of rapid variation is studied. A general strategy is introduced whereby mappings are adaptively constructed which map specified classes of rapidly varying functions into low order polynomials. A particular family of mappings constructed in this way is tested on a variety of rapidly varying functions similar to those occurring in applications. It is shown that the mapped function can be approximated much more accurately by Chebyshev polynomial approximations than the original function. The effect on the approximation of introducing subdomains is also studied. The accuracy of the pseudo-spectral approximation is very sensitive to the location of the interface, although this sensitivity is reduced when mappings are employed within the subdomains.

INTRODUCTION

One of the major difficulties in the application of Chebyshev pseudo-spectral methods, or other spectral methods, to the solution of partial differential equations is in the approximation of functions which exhibit localized regions of rapid variation. The approximation of such functions by polynomial expansions generally results in global oscillations unless a large number of terms are used in the polynomial approximation. These oscillations often lead to instabilities and inaccuracies when spectral methods are applied to the solution of partial differential equations [9, 11]. One possible remedy is to introduce a mapping to a new coordinate system so that the mapped function can be accurately approximated by low order polynomial expansions.

There is a great deal of computational evidence that appropriately chosen mappings can significantly enhance the accuracy of pseudo-spectral applications. In [18] mappings were shown to enhance the accuracy of approximations to shock like (hyperbolic tangent) functions. In [1, 3–5, 13] mappings were introduced dynamically via an adaptive procedure in which some error measure of the solution was minimized. Mappings have also been used to approximate boundary layer flows in Navier–Stokes calculations, for example [9].

There are two issues that must be addressed in the systematic application of mappings to enhance the accuracy of Chebyshev pseudo-spectral methods. These are

- the construction of an appropriate family of mappings;
- criteria to choose a particular mapping from this family according to the behavior of the solution to be approximated.

The second issue has been addressed by employing adaptive procedures in order to determine appropriate mappings [1, 3, 5, 13]. In these procedures, a family of mappings is introduced depending on a small number of free parameters. In addition, functionals are derived which are used to monitor the pseudo-spectral error. The appropriate mapping function is then chosen so that when the problem is transformed by the mapping the functional is minimized.

The choice of an appropriate functional to monitor the pseudo-spectral error is an important component of the adaptive procedure. In [5, 13] a Sobolev-type norm of the solution was chosen as the functional monitoring the pseudo-spectral error. In [3] another functional, derived

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directly from the Chebyshev expansion of an arbitrary function, was used in an adaptive procedure and shown to be more effective than the Sobolev norm in measuring the errors of the pseudo-spectral approximation.

Another equally important component of the adaptive procedure is the choice of the family of mappings to be utilized as coordinate transformations. Usually, the family of coordinate transformations to be incorporated in the solution method is prescribed in advance and justified by at most heuristic reasoning. There have been very few detailed studies of the effects of different families of mappings on the accuracy of pseudo-spectral approximations of various different types of functions. In [15] a family of mappings was derived which had the effect of giving a more uniform distribution to the collocation points, i.e., in an appropriate (but singular) limit the mapped Chebyshev collocation points were uniformly distributed. Numerical examples demonstrated that this mapping function gave enhanced accuracy compared to the unmapped Chebyshev pseudo-spectral method for functions which did not have rapid variations. However, the effect of the mapping on functions exhibiting rapid variation was not considered. A similar type of mapping was presented in [1]. In [3] a family of mappings was introduced which attempted to map certain interior regions to the boundaries.

In this paper we derive a family of mappings which are well suited to the approximation of functions with localized regions of rapid variations. The family of mappings depend on two parameters which are related to the size of the gradient over the narrow interval and the location of the region of rapid variation, respectively. The effect of this mapping on the accuracy of a Chebyshev approximation is analyzed for a variety of different functions and compared to the families of mappings in [3, 15] and to the case when no mapping is employed.

The use of mappings in the solution of partial differential equations has proven to be a very effective method in applying pseudo-spectral methods to problems with rapidly varying solutions. In this case the exact solution is not known and the functional which is to be minimized is evaluated using the last available solution. When the differential equation is transformed by a mapping, the coefficients of the differential operator change. The emphasis in this paper will be on the use of mappings to enhance the accuracy of the approximation of rapidly varying functions by Chebyshev pseudo-spectral methods. The effect of changes in the differential operator is not considered.

There is considerable computational evidence (see e.g. [1, 3, 4, 6, 13]) that the changes in the differential operator do not negatively effect the conditioning of the matrices obtained from the pseudo-spectral approximation. On the contrary, for some of the functions considered below, the use of mappings permits a reduction of almost an order of magnitude in the number of collocation points required for a given level of accuracy. The use of such a large number of collocation points would be prohibitively expensive and could lead to ill-conditioned matrices in forming the Chebyshev approximations. Similarly the computational results cited above do not indicate any severe reduction in the timestep by the use of mappings, and indeed the results of [15] indicate that for a fixed number of collocation points, the use of mappings can lead to larger timesteps than those that would be required if no mapping is employed. A comparison of the timesteps required for Chebyshev approximations must also account for the number of collocation points required for accurate approximations. The large number of collocation points required to accurately approximate rapidly varying functions without mappings will necessitate significantly smaller timesteps than those needed when mappings are employed with a consequently smaller number of collocation points. Furthermore, it is shown in [4] that standard iterative methods, for example, finite difference preconditioners, converge rapidly when adaptively chosen mappings are employed.

PSEUDO-SPECTRAL APPROXIMATION AND MAPPINGS

We first describe the derivation of the Chebyshev pseudo-spectral approximation. This discussion is brief, a more detailed description may be found in [9, 11]. Let $f(x)$ be a defined in the interval $I, -1 \leqslant x \leqslant 1$. The Gauss–Lobatto points are defined as

$$x_j = \cos\left(\frac{j\pi}{N}\right), \quad 0 \leqslant j \leqslant N. \tag{1}$$

P_n, the polynomial of nth degree which interpolates f at the points x_j, is the Chebyshev interpolant, or pseudo-spectral, approximation to f and can be obtained by using the Gauss–Lobatto quadrature rule to evaluate the expansion coefficients. It is known that the maximum norm error in approximating f by the pseudo-spectral approximation differs by at most a factor $O(\log N)$ from the maximum norm error in approximating f by the minimax polynomial [16]. A similar result is true for the Galerkin polynomial approximation to f. We anticipate that the results presented here are equally valid for Galerkin approximations.

In general a polynomial expansion is not appropriate to approximate functions with large gradients, e.g., functions exhibiting spikes, localized oscillations, or near discontinuities. Furthermore, the accuracy of Chebyshev approximations tends to be sensitive to the location of regions with large gradients. There is computational

evidence that Chebyshev approximations are more accurate for functions which vary rapidly near the boundaries as opposed to functions which vary rapidly away from the boundaries, for example [18]. It was shown in [18] that this is true both for Chebyshev methods, where the collocation points cluster near the boundaries, and also for collocation based on evenly spaced points. It appears from the results in [18] that this is a property of polynomial approximations and is not necessarily related to the clustering of the collocation points.

We denote the family of mappings by,

$$x = q(s, \alpha), \tag{2}$$

where x represents the physical coordinate, $-1 \leqslant s \leqslant 1$ is the transformed coordinate, and α denotes one or possibly more free parameters. The Chebyshev interpolant can then be used to approximate the transformed function $f(q(s, \alpha))$. The effect of the mapping can be regarded as transforming the function to be approximated to $f(q(s, \alpha))$ from $f(x)$. If the mapping is properly chosen $f(q(s, \alpha))$ will vary more gradually and so be more readily approximated by a polynomial. A different viewpoint is that in the original coordinate, x, we are changing the basis functions from polynomials to a new set of basis functions that can better represent functions that strongly vary in localized regions.

The mappings can improve the accuracy of the pseudo-spectral approximation in three general ways which can be employed as strategies in designing families of mappings. The transformed function can be less rapidly varying in the new coordinate system so it is better approximated by a polynomial. Second, the region of rapid variation can be mapped to the boundaries. Third, the mapping can expand the region near the boundaries thus tending to provide a more uniform distribution of collocation points. This latter approach was proposed in [15]. The distinction between these three potential strategies is not sharp, for example, a mapping in which a spike at $x = 0$ is transformed to a less rapidly varying function must of necessity map regions in the interior closer to the boundaries. However, in this paper we compare three families of mappings which were constructed based on these three potential strategies for several functions similar to those which occur in applications. Our results demonstrate that in most instances mappings based on transforming the function into one which varies more gradually tends to yield more accurate approximations.

When the Chebyshev pseudo-spectral method is used to solve partial differential equations there are many sources of errors and it is difficult to differentiate the source of the errors. In this paper we concentrate on the interpolation problem and evaluate the effectiveness of different mappings in reducing the maximum norm error in the pseudo-spectral approximation. In such a program one could simply start with a function and find the mapping parameters which yield the smallest error in some norm. However, in applications the exact solution is not known and some adaptive procedure is necessary in order to select appropriate mapping parameters. In this paper we determine mapping parameters by minimizing a functional related to the spectral interpolation error. This functional, derived in [3], is

$$I_2(g) = \left(\int_{-1}^{1} (\mathscr{L}^2 g)^2 / w(s)\, ds \right)^{1/2}, \tag{3}$$

where

$$w(s) = \sqrt{1 - s^2}, \qquad \mathscr{L} = w(s) \frac{d}{ds}.$$

If the interval $I, -1 \leqslant x \leqslant 1$ is mapped to the interval $0 \leqslant \theta \leqslant \pi$ by the mapping $x = \cos(\theta)$, then (3) is transformed to the integral of the square of the second derivative of the transformed function. This was demonstrated in [12] where it was also shown that (3) provides an upper bound on the weighted L_2 norm of the projection error (i.e., the error obtained from approximating u by its Chebyshev Galerkin approximation). Furthermore, this functional gives an upper bound on the maximum norm of the error [3].

Our approach is to compute (3) using the transformed function $f(q(s, \alpha))$. We do this by forming the Chebyshev interpolant to $f(q(s, \alpha))$ using a fixed value of N and then, using the Gauss–Lobatto quadrature formula, evaluating (3). The result will be a function only of the parameter vector α which we denote by $I_2(\alpha)$. We choose α so as to minimize I_2. In practice this minimum is found by computing $I_2(\alpha)$ over a range of values of α and then choosing that parameter for which a minimum occurs. In most cases we take the global minimum over the range of values of α. In certain circumstances global minima occur for parameters which give a highly inaccurate approximation. This is due to inaccuracies in computing the quadrature in (3) when the transformed function $f(q(s, \alpha))$ is poorly approximated for some parameters α. In this case we take local minima for which the Chebyshev interpolant is a good approximation to the given function f.

This approach is a practical and effective method to determine mappings which enhance the accuracy of the pseudo-spectral approximation [3]. It has been successfully utilized in the computation of problems in combustion [3, 5, 13] and has also been used for problems in wave propagation in [1]. Furthermore in all cases presented below the errors that were obtained from mappings for which $I_2(\alpha)$ was minimized were very close to the minimum errors in the maximum norm over the range of parameters investigated.

3. FAMILIES OF MAPPINGS

We first introduce a family of mappings which have the property of transforming functions of a certain class to polynomials. Although the functions occurring in applications are not exactly of the specified class, they would be expected to be sufficiently similar so that the Chebyshev approximation requires only low degree polynomials to give good accuracy.

We begin with a specified class of functions of the form $s = h(x, \alpha_1, \alpha_2)$. We suppose for certain parameter values this function exhibits rapid variations. For example, suppose that for large values of α_1, this function exhibits shock-like behavior with the rapid variations occurring near the point $x = \alpha_2$ and nearly constant behavior away from this point. If the function $h(x, \alpha_1, \alpha_2)$ is univalent then upon an appropriate rescaling this function can be assumed to map the interval $-1 \leqslant x \leqslant 1$ univalently onto itself. The inverse function $x = h^{-1}(s, \alpha_1, \alpha_2)$ then describes a family of mappings such that under this mapping the function h becomes linear and can therefore be approximated by a low order expansion of Chebyshev polynomials. In applications the rapidly varying solutions will not be exactly of the form of the given function $h(x, \alpha_1, \alpha_2)$, however, the image of these functions under the mapping given by h^{-1} is likely to have a gradual variation and so can be approximated by a low order polynomial expansion provided the parameters α_1 and α_2 are properly chosen. This will be justified by the examples presented below.

The mapping $x = h^{-1}(s, \alpha_1, \alpha_2)$ can be expected to enhance the approximation not only for rapidly varying functions similar to $h(x, \alpha_1, \alpha_2)$ but also for rapidly varying functions which behave like derivatives of $h(x, \alpha_1, \alpha_2)$. In order to see why this is so, let $g = h'(x, \alpha_1, \alpha_2)$ where $'$ denotes derivative with respect to x. If $h(x, \alpha_1, \alpha_2)$ behaves as a step function then $g(x, \alpha_1, \alpha_2)$ will behave as a spike centered at $x = \alpha_2$. Suppose that this point is located away from the boundaries. Using the chain rule we obtain an expression for $g(x, \alpha_1, \alpha_2)$ under the mappings $s = h(x, \alpha_1, \alpha_2)$,

$$g(h^{-1}(s, \alpha_1, \alpha_2), \alpha_1, \alpha_2) = (\partial h^{-1}(s, \alpha_1, \alpha_2)/\partial s)^{-1}. \quad (4)$$

If the function $h(x, \alpha_1, \alpha_2)$ behaves as a step function changing rapidly near the interior point $x = \alpha_2$, then clearly $\partial h^{-1}(s, \alpha_1, \alpha_2)/\partial s$ will change most rapidly near the boundaries in s. Therefore we expect the mapping to enhance the accuracy of the approximation for g. Similar arguments hold for the approximation of functions similar to higher derivatives of $h(x, \alpha_1, \alpha_2)$.

In this paper we consider the following function $h(x, \alpha_1, \alpha_2)$

$$s = s_0 + \tan^{-1}(\alpha_1(x - \alpha_2))/\lambda. \quad (5)$$

For large values of α_1 this function is nearly discontinuous with a region of rapid variation occurring near $x = \alpha_2$. The parameters s_0 and λ are determined so that (5) maps the interval I onto itself. The values of s_0 and λ, determined in this way are

$$s_0 = \frac{\kappa - 1}{\kappa + 1}, \quad \kappa = \tan^{-1}(\alpha_1(1 + \alpha_2))/\tan^{-1}(\alpha_1(1 - \alpha_2)),$$

$$\lambda = \tan^{-1}(\alpha_1(1 - \alpha_2))/(1 - s_0).$$

With these choices of λ and s_0 the inverse of (5),

$$x = \alpha_2 + \tan((s - s_0)\lambda)/\alpha_1, \quad (6)$$

describes a two-parameter family of mappings of I into itself which is suitable for the accurate resolution of functions with localized regions of rapid variation. Here α_1 is related to the degree of change of the function and α_2 is related to the location of the region of rapid variation. In applications these parameters would be obtained either from prior knowledge of the solution or from minimizing a functional such as (3) which measures the error in the approximation. We note that (6) is explicitly invertible.

For small α_1, s is approximately equal to x and (6) is approximately the identity map. The use of the mapping (6) should not therefore be expected to degrade the accuracy in approximating slowly varying functions which can be well approximated by a Chebyshev approximation without any mapping.

We point out that the an alternative to the inverse tangent function is the hyperbolic tangent function with an appropriate scaling. Mappings constructed using the hyperbolic tangent are related to stationary shock-like solutions of the Burgers equation [7].

An alternative family of mappings has been proposed in [15]. This family of mappings is

$$x = \arcsin(\alpha_1 s)/\arcsin(\alpha_1) \quad (7)$$

and $0 \leqslant \alpha_1 \leqslant 1$. The effect of this mapping is to expand the boundary regions and compress the interior regions. As $\alpha_1 \to 1$ the image of the Gauss–Lobatto points become uniformly spaced, however, the mapping becomes singular. As $\alpha_1 \to 0$, (7) approaches the identity mapping. The analysis and examples presented in [15] demonstrate that the use of (7) with α_1 near 1 can significantly improve the accuracy of pseudo-spectral approximations for a range of functions. This was attributed to fact that the collocation points become nearly uniformly spaced as $\alpha_1 \to 1$. The major effect of this mapping is to obtain a more uniform distribution of the collocation points and as such the mapping depends on only one parameter (however, a two-parameter extension was proposed in [15]). A related mapping is

presented in [1] which permits a concentration of collocation points in the interior which exceeds that at the boundary, i.e., it goes beyond obtaining a uniform distribution of the collocation points.

A third family of mappings that we consider is

$$s = 4/\pi \tan^{-1}[\alpha_1 \tan(\pi/4)(x' - 1)] + 1 \tag{8}$$

with

$$x' = (\alpha_2 - x)/(\alpha_2 x - 1).$$

The use of this mapping is described in [3, 5]. The effect of the mapping is to map an arbitrary point, α_2, to the origin and then expand one of the boundaries at the expense of the other by varying α_1. This mapping is presented in [5], where it was used with $\alpha_2 = 0$ to compute functions with rapid variations near the boundary and in [3] to compute functions with rapid variations in the interior.

4. MAPPINGS AND ACCURACY

In this section we compare the effectiveness of the families of mappings described by (6), (7), and (8) in enhancing the accuracy of pseudo-spectral approximations. Our approach is to construct the pseudo-spectral polynomial for the transformed function and then to measure the maximum norm of the error. The maximum norm of the error is computed by comparing the approximating polynomial and the given function over a large grid of points. We also compute the discrete L_2 (unweighted) norm of the error in the coordinate system selected. We note that the L_2 norm of the error depends on the coordinate system and may not adequately measure the relevant errors in applications.

We first consider approximations to the function

$$y = \tanh(\sigma(x - x_0)). \tag{9}$$

TABLE I

$y = \tanh(\sigma(x - x_0))$

Mapping	α_1	α_2	I_2	Max error	L_2 Error
6	51.30	0.0	2.40	1.71e-14	1.10e-14
6	6.06	0.2	80.57	8.37e-05	2.18e-05
7	0.99999	NA	132.54	1.51e-03	3.48e-04
7	0.9	NA	181.24	6.72e-03	1.42e-03
8	9.92	−0.887	189.12	7.92e-03	1.70e-03
U	NA	NA	227.14	2.05e-02	4.02e-03

with $\sigma = 40$. In our computations we compute over a grid of values of α_1 and α_2 and select those parameter values for which the functional I_2 is minimized. For the case of (6) the global minimum always occurs when $\alpha_2 = x_0$, however, we indicate the sensitivity of the approximation to the value of α_2. The approximating polynomial is computed using $N = 121$. The error is computed using the Gauss–Lobatto points with $N = 351$. Both the maximum norm and the discrete L_2 norm of the error are computed. The results are presented in Table I. An entry of U in the first column denotes that no mapping was used.

The data in this table illustrates the effectiveness of the mapping (6) in enhancing the approximation of the shock-like profile (9). A graphical illustration of this is shown in Figs. 1a and 1b. In Fig. 1a we plot (9) against the original independent variable x. In Fig. 1b we plot the transformed function against the variable s using $\alpha_1 = 51.30$ and $\alpha_2 = 0$. The figures clearly illustrate the more gradual variation of the transformed function.

We note that the mapping (7) allows for a more accurate approximation than if no mapping at all is employed and also permits a considerably more accurate approximation than the use of (8). The best accuracy (and the minimum value of (3)) occurs as $\alpha_1 \to 1$ so that the collocation points are uniformly spaced but the mapping is singular. Values of

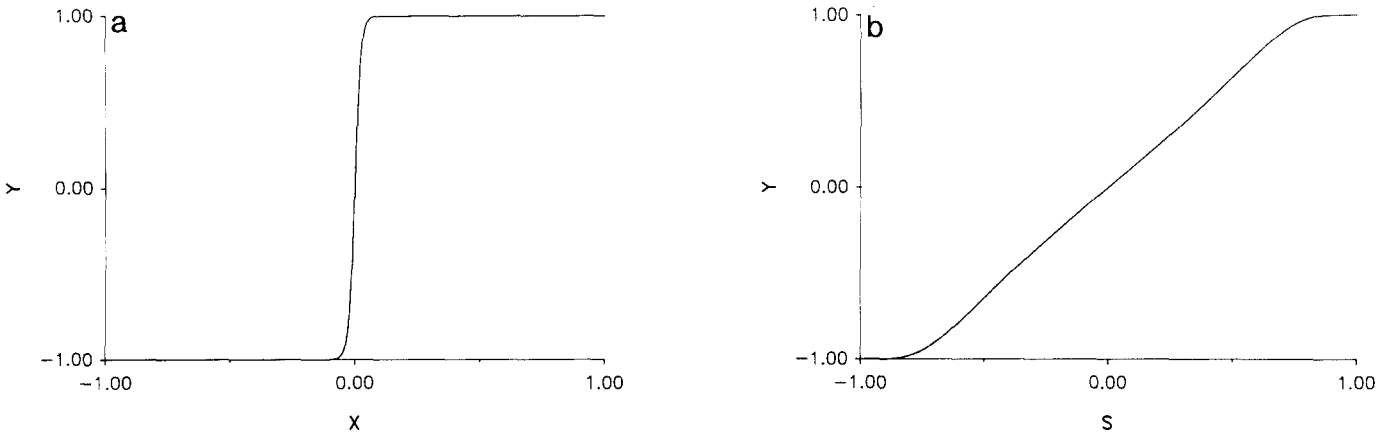

FIG. 1. (a) Function (9) plotted against the independent variable x and (b) plotted against the transformed variable s.

α_1 close to 1 do not affect our approximation program, however when mappings are applied to the solution of partial differential equations, Jacobians are introduced into the equations. The singularity in the mapping (7) which occurs at $\alpha_1 = 1$ may therefore effect the solution of partial differential equations when α_1 is sufficiently close to 1. In the solution of partial differential equations the degree of "near singularity" that can be allowed in the mapping without affecting the numerical approximation will be problem dependent, depending on other parameters of the solution process not evaluated here. There is a degradation in accuracy as α_1 is reduced from the singular value 1. We have presented results for $\alpha_1 = 0.9$ in order to illustrate the behavior of the approximation as α_1 is varied.

Finally we note that (6) is robust in the sense that even if α_2 is not at the point of maximum variation it still yields a significantly improved approximation. This is important for several reasons.

- The minimization might be inexact.
- There may be several regions of rapid variation clustered together.
- Different dependent variables might exhibit rapid spatial variation at different points.
- In many two-dimensional problems there is often a principal direction in which the spike (or rapid variation) occurs, however, the location of this variation may depend on some transverse coordinate. It may be more efficient to employ a one-dimensional mapping to resolve such a solution rather than a two-dimensional mapping which depends on the transverse coordinate.

The behavior described in the last point above is common in, for example, wave propagation if a pulse propagates through a dispersive medium, fluid dynamics if curved shocks form and combustion where cells can form along flame fronts. Thus we consider the robustness of the mapping (6) with respect to the location parameter α_2 to be an important point.

We next illustrate the convergence properties for the mapped function by considering the same results using $N = 81$. The results are presented in Table II. In this case we have used the same mapping parameters as were obtained using 121 collocation points. The results show that there is a much more rapid convergence when the mapping (6) is used. This is probably due to infinite order convergence of the spectral method which in practice means a very high order of convergence when the function varies gradually. We note that there are definite numerical errors in the approximation of the function I_2 when the mapping (6) is not used. In practice this could lead to poor parameter values being produced by the minimization procedure.

We have also attempted to compare the number of collocation points required for comparable accuracy between approximations which employ the mapping (6) and those for which no approximation is employed. Using 51 collocation points the use of the mapping (6) with adaptively chosen parameters results in a maximum error of about 2.5×10^{-8}. If no mapping is employed we find that 450 collocation points are required to obtain a comparable error (4.8×10^{-8}). Similar comparisons also hold for higher error levels. For example, if the mapping (6) is employed, 20 collocation points are required in order to obtain an error of the order 6.8×10^{-5}, while if no mapping is employed 270 points are required for a comparable error (5.9×10^{-5}).

The results described for (6) are not necessarily optimal as the family of mappings (6) is constructed so that the image of a near step function is linear. We next consider the function

$$y = \exp(-\sigma^2(x - x_0)^2/2). \tag{10}$$

This function also exhibits a rapid variation near $x = x_0$, however, the behavior is that of an isolated spike rather than that of a near step as in (9). We consider the values $\sigma = 50$ and $x_0 = 0$. The results are presented in Table III.

We can infer from the results presented in Table III that the mapping (6) is effective in enhancing the resolution of spike-like functions. We also illustrate this graphically in Figs. 2a and 2b where we plot the function (10) in the original coordinate, x (Fig. 2a), and the transformed function in the mapped coordinate, s (Fig. 2b). We note that the

TABLE II

$y = \tanh(\sigma(x - x_0))$, Using $N = 81$

Mapping	α_1	α_2	I_2	Max error	L_2 Error
6	51.30	0.0	2.40	2.20e-11	1.70e-11
6	6.06	0.2	80.95	1.96e-03	6.68e-04
7	0.99999	NA	119.22	1.60e-02	4.08e-03
8	9.92	−0.887	137.60	5.11e-02	1.12e-02
U	NA	NA	130.77	8.68e-02	1.92e-02

TABLE III

$y = \exp(-\sigma^2(x - x_0)^2/2)$, $x_0 = 0$

Mapping	α_1	α_2	I_2	Max error	L_2 Error
6	37.48	0.0	18.26	1.42e-09	8.30e-10
6	10.86	0.1	106.67	3.37e-06	1.48e-06
6	5.744	0.2	245.51	1.59e-03	5.51e-04
6	3.506	0.3	386.86	2.20e-02	6.08e-03
7	0.99999	NA	401.10	1.20e-02	6.06e-03
7	0.9	NA	557.02	4.90e-02	1.96e-02
8	73.92	−0.983	547.04	4.57e-02	1.85e-02
U	NA	NA	699.84	1.14e-01	3.95e-02

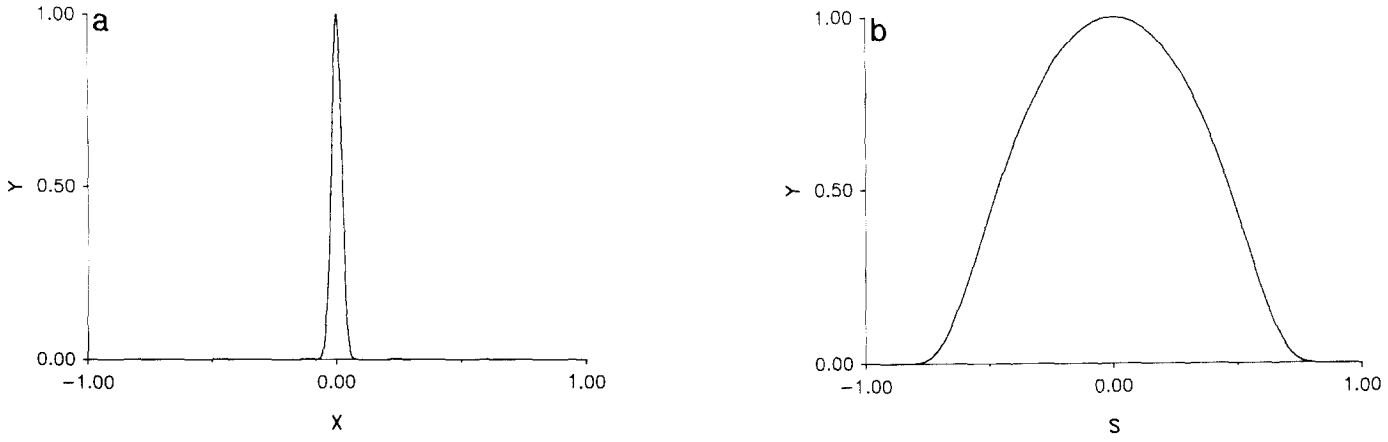

FIG. 2. (a) Function (10) plotted against the independent variable x and (b) plotted against the transformed variable s.

mapped function still resembles a Gaussian, however, there is a more gradual variation. The use of mappings for the function (10) also leads to a significant reduction in the number of collocation points required in order to obtain a given level of accuracy. For example, if the mapping (6) is employed, 31 collocation points are required for a maximum norm error of the order of 8.9×10^{-5}. If no mapping is employed, 200 collocation points are required for a comparable error (6.4×10^{-5}).

The results also illustrate that the use of the mapping (6) provides good accuracy as the parameter α_2 is moved away from the location of the spike. The results suggest that the mapping (6) can allow high resolution of multiple spikes even when the parameter α_2 deviates significantly from the location of each individual spike. Of course if the spikes are widely separated or if α_2 is far from the center of the spike the effectiveness of (6) degrades. We have found that the accuracy obtained from using (6) does not strongly depend on the value of x_0 provided α_2 is chosen to be near x_0. Similar results hold for functions exhibiting a step function type of behavior as in (9). We also note that the mapping (7) provides considerably more accuracy than (8).

The results in Table III indicate that the functional (3) has a rather sharp minimum when α_2 is at the location of the spike. In our calculations we have found the value of α_2 obtained from minimizing (3) to be an excellent indicator of the location of the spike (or most rapid variation). This suggests that this procedure could be used as a shock locator for non-oscillatory spectral methods and filtering methods [8, 10], at least when only one shock is present or multiple shocks are closely spaced.

The effect of the location of the spike does considerably effect the behavior obtained from using (7) and (8). We illustrate this in our next example where we consider a spike near the boundary by setting $x_0 = 0.9$. The results are presented in Table IV.

The mapping (6) still gives very good accuracy even when α_2 is not close to x_0. The best approximation using (7) now occurs when $\alpha_1 \to 0$ where the mapping approaches the identity. In Table IV we indicate the errors found for $\alpha_1 = 0.0001$ the limit of our search region, together with the errors for larger values of α_1, which lead to approximately uniformly spaced points. For functions which have significant variations near the boundaries, having the collocation points more uniformly spaced leads to a degradation in accuracy. We find that the behavior of the mapping (7) depends very abruptly on the location of the spike. For values of x_0 just below 0.8 we find an abrupt transition where the value of α_1 yielding maximum accuracy switches from α_1 near 1 to α_1 near 0. In this narrow overlap region the results obtained from (7) are insensitive to α_1.

The result using (8) were found to be insensitive to α_2 for α_2 near 0, and are presented only for the mapping with $\alpha_2 = 0$ as it is used in [5] for functions with rapid variation near the boundary. The results indicate that this mapping is effective in approximating boundary layer type solutions however it does not seem to offer any advantages over (6) and is not considered further.

TABLE IV

$y = \exp(-\sigma^2(x-x_0)^2/2)$, $x_0 = 0.9$

Mapping	α_1	α_2	I_2	Max error	L_2 Error
6	35.25	0.9	15.05	3.29e-10	2.21e-10
6	1.89	0.5	179.55	1.29e-04	5.10e-05
7	0.0001	NA	227.22	1.01e-03	4.25e-04
7	0.9	NA	288.03	2.85e-03	1.48e-03
7	0.99999	NA	400.23	1.34e-02	5.81e-03
8	0.0896	0.0	54.97	8.31e-10	4.03e-10
U	NA	NA	227.22	1.01e-03	4.25e-04

TABLE V

Sum of Two Gaussians, $x_0 = 0.30$, $x_1 = -0.15$

Mapping	α_1	α_2	I_2	Max error	L_2 Error
6	5.136	0.0700	386.20	2.10e-03	1.30e-03
7	0.99999	NA	567.2	1.56e-02	8.17e-03
7	0.90000	NA	787.1	7.81e-02	2.60e-02
U	NA	NA	1031.0	1.70e-01	5.54e-02

We next consider the function

$$y = \exp(-\sigma^2(x - x_0))^2/2) + \exp(-\sigma^2(x - x_1))^2/2). \quad (11)$$

The objective will be to determine the behavior of the various mappings when there is more than one region of rapid variation. The separation of the two regions of rapid variation can be controlled by varying x_0 and x_1. We first consider the case $x_0 = 0.30$, $x_1 = -0.15$, $\sigma = 50$. The results are presented in Table V.

It can be seen from the table that the family of mappings described by (6) yields a more accurate Chebyshev approximation even in the case of relatively separated spikes. We note that the minimum error with the mapping (7) again appears to occur when $\alpha_1 \to 1$ so that the collocation points are nearly uniformly spaced but the mapping is almost singular. We have also exhibited the degradation in accuracy obtained by reducing α_1 from 1 by specifying an upper limit on α_1 of 0.9. Even with this degradation the results obtained using (7) are considerably better than for the case when no mapping is employed.

We next consider more widely separated spikes by setting $x_0 = 0.5$, $x_1 = -0.25$, $\sigma = 40$. The results are presented in Table VI. This is a relatively severe case as neither of the spikes are located close to the boundary. We observe that for widely separated spikes located away from the boundaries the use of (6) leads to larger errors than (7), provided values of α_1 near 1 are used. The results obtained from (7) are not strongly sensitive to x_0. This is not surprising as the effect of the mapping is to make the collocation points more uniform (in x).

TABLE VI

Sum of Two Gaussians, $x_0 = 0.50$, $x_1 = -0.25$

Mapping	α_1	α_2	I_2	Max error	L_2 Error
6	2.704	0.0780	625.6	4.59e-02	1.43e-02
7	0.99999	NA	567.1	1.45e-02	7.96e-03
7	0.90000	NA	758.8	5.76e-02	2.35e-02

TABLE VII

Sum of Two Gaussians, $x_0 = 0.90$, $x_1 = 0.2$

Mapping	α_1	α_2	I_2	Max error	L_2 Error
6	3.172	0.361	375.3	3.86e-03	1.57e-03
7	0.99999	NA	566.6	1.43e-02	7.71e-03
7	0.90000	NA	629.3	8.01e-02	1.99e-02
U	NA	NA	742.91	1.44e-01	3.78e-02

We finally consider the case where one of the spikes is located near the boundary. In this case we consider $x_0 = 0.9$, $x_1 = 0.2$, $\sigma = 50$. The results are presented in Table VII. We note that the best approximation occurs when α_2 is close to the location of the interior spike.

In the case of multiple regions of rapid variation the optimum values of α_1 and α_2 for the mapping (6) are not suggested by the locations of the regions of rapid variation. These parameters must be determined by a numerical minimization procedure.

In summary, our results indicate that the strategy of transforming the given function to one that is more readily approximated by a polynomial, which leads to the derivation of the family of mappings (6) appears to be more effective than the strategy of mapping regions of rapid variation to the boundaries. In particular the use of (6) appears to be superior in all respects to (8). The parameter α_2 in (6) is an excellent indicator of the location of maximum variation, but the mapping is very robust if α_2 is varied away from this location.

When the rapid variation occurs away from the boundary the use of (7) with α_1 near 1, so that the collocation points are nearly uniformly spaced, is preferable to the unmapped Chebyshev method. In these cases the mapping (7) is most accurate as $\alpha_1 \to 1$ and degrades when α_1 is reduced from the singular value. It is therefore preferable to use this mapping as close to the singular point as possible. When there is a single region of rapid variation near the boundaries the unmapped Chebyshev approximation is more accurate than the use of (7).

The use of (6) is superior to (7) for functions with rapid variation except when there are multiple regions of rapid variations that are widely separated and not located near the boundaries. In this case the best strategy seems to be to use (7) to obtain nearly equally spaced collocation points. Although this analysis is presented for pulse like functions similar effects hold for other types of functions with regions of rapid variation, for example functions which exhibit a step function type of behavior or localized highly oscillatory behavior.

In this paper we have concentrated on the role of

TABLE VIII

$y = 0.05/(x^2 + 0.05)$

Mapping	α_1	α_2	I_2	Max error	L_2 Error
6	6.54	0.0	2.68	1.14e-12	9.69e-13
7	0.99624	NA	7.99	3.66e-05	3.06e-04
U	NA	NA	14.59	7.80e-04	5.36e-04

TABLE X

Burgers Equation varying N

m	πt	α_1	N
151.7508	1.61622	37.0	33
152.3011	1.60446	37.0	49
152.3214	1.60404	37.0	61

originally introduced to decrease the spectral radius of the spectral differentiation operator and thereby increase the allowable timestep in the solution of time dependent partial differential equations. This is not considered here.

Although the major emphasis in this paper is on the approximation of rapidly varying functions we have found that the mapping (6) is also effective in approximating functions which vary more gradually. We illustrate this with the example

$$y = 0.05/(x^2 + 0.05), \tag{12}$$

which was also considered in [15]. In this case we use 32 collocation points. The results of the approximation are shown in Table VIII.

We next illustrate the effectiveness of this mapping in approximating solutions of partial differential equations. We consider Burgers equation

$$u_t + uu_x = vu_{xx}, \tag{13}$$

where $-1 \leqslant x \leqslant 1$. The initial condition and the boundary conditions are chosen as

$$u(x, 0) = -\sin \pi x, \qquad u(\pm 1, t) = 0.$$

The viscosity coefficient $v = 0.01/\pi$. This problem has been used as test case for a variety of spectral, pseudo-spectral and finite difference methods [2, 13]. As t increases the solution develops a very steep gradient at $x = 0$. The maximum gradient at $x = 0$ occurs when πt is close to 1.6. As t increases beyond this point the solution relaxes to zero. We have solved this problem using an adaptive pseudo-spectral method together with the mapping (6). The solution is symmetric about $x = 0$ and the minimum of (3) always occurs at $\alpha_2 = 0$.

The solution to this problem can be computed analytically. One measure of accuracy, which has been used in [2, 13], is the quantity

$$m = \max(u_x(0, t)).$$

The analytic result is $m = 1.5200516$ occurring at $\pi t = 1.6037$ [2]. We have found, analogously to what was found in [13], that the coordinate system which yields the most accurate calculation of m is not perfectly predicted by the adaptive procedure. Minimizing the functional (3) appears to concentrate too much resolution near $x = 0$ at the expense of resolution away from $x = 0$. A similar conclusion was arrived at in [13] using an adaptive procedure based on minimizing a different functional. When the search region is artificially constricted so that the parameter α_1 does not get too large, effectively reducing the resolution near $x = 0$, we find excellent agreement with the analytic value for m. We indicate the computed values of m in Table IX. The number of collocation points, N, was 61. The first entry in the table is the result when the value of α_1 was restricted, while the second entry is for the value of α_1 selected by the adaptive procedure.

Using the mapping (6), a high degree of accuracy can be attained as the number of collocation points is reduced. We indicate this in Table X, where we list the values of m obtained for different numbers of collocation points. In this table we have not restricted the search for α_1 but rather taken the value of α_1 predicted by minimizing (3). In most calculations the exact solution is not available and the minimization of some functional is at present the only practical method to predict the appropriate coordinate system.

TABLE IX

Burgers Equation with $N = 61$

m	πt	α_1
151.9896	1.60407	21.5
152.3214	1.60404	37.0

5. MULTIPLE DOMAINS

The results presented above indicate that the mapping (6) can significantly improve the Chebyshev approximation to rapidly varying function when there is only one region of rapid variation or when there are several closely spaced regions of rapid variation. When there are widely spaced regions of rapid variation away from the boundaries the effectiveness of (6) degrades and the best strategy within the

context of a single domain computation is to use (7) with α near 1 so as to obtain a nearly uniform distribution of collocation points. However, the accuracy obtained from using (7) is generally less than the accuracy that could be obtained from use of the mapping (6) if the regions of rapid variation were not widely separated.

One possible approach to this problem is to extend the mapping (6) to allow for the resolution of multiple regions of rapid variation. Such an extension would require additional parameters in the mapping, and therefore the resulting minimization problem to determine appropriate parameters would be considerably more expensive than for the case when the mapping depends on only two parameters.

An alternative approach is to introduce two or more subdomains and employ mappings whithin each subdomain. The use of multiple domains can lead to an improvement in accuracy by (a) resolving small scale structures in the problem by introducing domains corresponding to the length scales appropriate to the problem, (b) choosing the interface so that the small scale structures or rapid variations occur near the boundary, and (c) isolating different regions of rapid variations within each subdomain and then employing mappings such as (6) within each subdomain in order to improve the accuracy of the unmapped Chebyshev approximation.

Multiple domain procedures have other advantages in addition to possible improvements in accuracy. For example, they lend themselves to parallel computation and can lead to smaller and better conditioned matrices, see for example [9, 14, 17]. In this paper, however, we only consider the effect of multiple domains on the accuracy of approximation of rapidly varying functions. In particular we demonstrate that a strategy such as that described by point (c) above, involving the interaction of mappings and domain decomposition, can lead to significant improvements in the accuracy of Chebyshev interpolation.

An alternative strategy is to employ multiple domains without any mappings. This strategy would be attractive if the effect of the mappings was to introduce an ill conditioning into the discretized differential operator. However, no such ill conditioning has yet been found.

In the context of the solution of partial differential equations, the use of multiple domains introduces additional parameters which have to be determined, in particular the number of domains, the location of the interface points, and the number of collocation points within each subdomain. If the multiple domains are chosen to correspond to very small scale structures or localized rapid variations, then the accuracy of the resulting approximation may be quite sensitive to the location and extent of the interface points. These parameters are generally chosen non-adaptively or with prior knowledge of the behavior of the function to be approximated. A general solution procedure for problems with rapid solutions would require adaptive procedures to determine both the number of the subdomains and their locations.

When mappings are employed additional parameters must be determined. For example, if there are two domains and the mapping (6) is employed in each subdomain then there are seven parameters which have to be determined (the parameters of the mapping, the location of the interface and the number of collocation points within each subdomain). In this paper we constrain the number of points so that each domain has an equal number of points. This strategy is preferable for parallel computation, although it may not necessarily be preferable for determining the most accurate approximation with a minimum number of computational degrees of freedom. We will also constrain the number of domains to be two. We now require the determination of five parameters, including in particular some procedure to determine the location of the interface, e.g., some functional which measures the error as a function of the location of the interface. The determination of these parameters could be significantly simplified if the location of the interface could be determined independently of the mapping parameters within each subdomain, i.e., a strategy whereby the location of the interface is determined from the current solution, and then new mapping parameters are determined within each subdomain by the minimization procedure described above. We demonstrate that a strategy of employing multiple domains to isolate regions of rapid variation, together with the mapping (6) to resolve localized regions of rapid variation within each subdomain can lead to such a decoupling of the problem of determining these parameters.

In order to accomplish this it is first necessary to understand the role of the location of the interface on the accuracy of the approximation and to develop functionals which can monitor this. We therefore first consider the use of multiple domains without any mapping within each subdomain and then demonstrate that once an appropriate interface is found the errors in the approximation can be further reduced by employing mappings such as (6) within each subdomain. We consider the approximation of (11) with $\sigma = 50$ and with 41 collocation points in each subdomain. We further restrict ourselves to the case where $x_0 = 0.5$, $x_1 = -0.25$, although similar results have been obtained for other values of these parameters. The accuracy of single domain Chebyshev approximations to this function under various mappings is described above. We now describe the effect on accuracy of breaking up the original domain into two subdomains. We compute the largest of the maximum norm errors within the two subdomains.

The location of the interface, Q, will be obtained adaptively by determining Q so that the maximum of (3) in each subdomain is minimized without using any mapping. Since we use an equal number of points in each subdomain, this

TABLE XI

Effect of Mappings on Multi-domain Approximation

I_2^1	I_2^2	Q	Mapping	α_1^1	α_2^1	α_1^2	α_2^2	Max error
1018.0	Single domain		U	NA	NA	NA	NA	1.95e-01
625.6	Single domain		(6)	2.704	0.078	NA	NA	4.59e-02
567.1	Single domain		(7)	0.99999	NA	NA	NA	1.45e-02
316.6	219.2	0.125	U	NA	NA	NA	NA	2.18e-01
17.1	16.9	0.125	(6)	16.93	0.331	13.25	−0.146	4.23e-06
246.7	232.1	0.083	U	NA	NA	NA	NA	1.23e-01
17.25	16.9	0.083	(6)	15.7	0.386	13.25	−0.0908	5.02e-06
158.1	123.8	0.083	(7)	0.99999	NA	0.99999	NA	3.68e-02

corresponds to minimizing an estimate for the largest of the maximum norm errors. In Table XI we show that this strategy leads to an improvement in accuracy over the single domain result and in the case where the interface is chosen in an ad hoc manner to be equidistant from both spikes, i.e., $Q = (x_0 + x_1)/2$. We have then employed the mapping (6) within each subdomain to determine parameters α_1^1, α_2^1, α_1^2, α_2^2, which minimize (3) in each subdomain as in Section 4 above (we employ a simple linear mapping to map each subdomain to the interval I).

It is apparent from Table XI that the overall accuracy of the approximation is very sensitive to the location of the interface when no mapping is employed. In this case the strategy of minimizing the maximum, over the two domains, of (3) leads to a significant reduction of the error.

The use of the mapping (6) can lead to a dramatic reduction in the error when there are multiple domains. In this case the overall accuracy of the Chebyshev interpolation is not sensitive to the location of the interface as long as the domains are such that there are not widely separated regions of rapid variation within each domain. In the solution of partial differential equations it may be important to determine the domains adaptively so that the regions of rapid variation are localized within each subdomain. The strategy of minimizing the maximum of (3) appears to accomplish this. Although we illustrate this behavior for only one example we have observed this behavior for other cases, not reported here, as well.

The use of (7) leads to an improvement in accuracy over the unmapped Chebyshev approximation but the improvement is not as great as in the case of the single domain. We note that the results obtained using (7) are slightly worse than in the unmapped single domain case. This may be due to the fact that the collocation points are nearly uniformly spaced and since $Q \neq 0$, the spacing of the c
is greater in one of the subdomains. Setti
errors very similar to the single domain ca

In summary, the results indicate that the
ping (6) in conjuction with a multiple do
can lead to an improvement in Chebyshev i
pseudo-spectral methods for the case that t
approximated has multiple regions of
Furthermore this improvement can be obta
tive procedures without prior knowledge o
the rapid variation.

Note added in proof. Boyd [19] has recently inves
mappings related to (8) which increase resolution in
domain.

REFERENCES

1. B. Fornberg, *Math. Comput.* **51**, 699 (1988).
2. C. Basdevant *et al.*, *Comput. Fluids* **14**, 23 (1986).
3. A. Bayliss, D. Gottlieb, B. J. Matkowsky, and M. Mink
Phys. **81**, 421 (1989).
4. A. Bayliss, R. Kuske, and B. J. Matkowsky, *J. Comput.*
(1990).
5. A. Bayliss and B. J. Matkowsky, *J. Comput. Phys.* **71**, 147
6. A. Bayliss, B. J. Matkowsky, and M. Minkoff, *SIAM J. App*
1421 (1989).
7. J. M. Burger, *The Nonlinear Diffusion Equation* (Reidel, Bost
8. W. Cai, D. Gottlieb, and C. Shu, *Math. Comput.* **52**, 389 (198
9. C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang,
Methods in Fluid Dynamics (Springer-Verlag, New York, 1987)
10. D. Gottlieb, in *Proceedings, 9th International Conference on Num*
Methods in Fluid Dynamics; edited by Soubbarameyer and Bo
(Springer, New York, 1985), p. 48.
11. D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Metho*
Theory and Applications (C.B.M.S.–N.S.F. Conference Series i
Applied Mathematics, SIAM, Philadelphia, 1977.
12. H. Guillard, J. M. Male, and R. Peyret, University of Nice, preprint
no. 270, 1990, to appear, *J. Comput. Phys.*
13. H. Guillard and R. Peyret, *Comput. Methods Appl. Mech. Eng.* **66**, 17
(1988).
14. D. A. Kopriva, *SIAM J. Sci. Stat. Comput.* **10**, 120 (1989).
15. D. Kosloff and H. Tal-Ezer, ICASE Report No. 89–71 (1989), to
appear, *J. Comput. Phys.*
16. J. H. McCabe and G. M. Phillips, *BIT* **13**, 434 (1973).
17. M. G. Macareg and C. L. Streett, *Appl. Numer. Math.* **2**, 95 (1986).
18. A. Solomonoff and E. Turkel, *J. Comput. Phys.* **81**, 239 (1989).
19. J. P. Boyd, *J. Comput. Phys.* **98**, 181 (1992).